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Computational Statistics

The Evaluation of Moments of Ratios of Quadratic Forms in Normal Variables and Related Statistics (QRMOM): Technical Description

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SUMMARY
This paper considers ratios of quadratic forms, $R = x'Ax/x'Bx$, where x is normally distributed with mean μ and variance-covariance matrix Ω (positive definite), A is symmetric and B positive semi-definite. The subroutine QRMOM can calculate for $s \geq 1$, $E[R^s], E[R^s(a'x)]$ and $E[R^s(x'Cx)]$, where a is a vector and C a symmetric matrix of appropriate order. QRMOM also checks the existence of the moments.

Keywords: Ratios of quadratic forms; Moments; Calculation of expectations; Tests for existence.

1 Introduction

We consider ratios of two quadratic forms, $R = x'Ax/x'Bx$, where x is normally distributed with mean μ and variance-covariance matrix Ω (positive definite), A is symmetric and B is symmetric positive semi-definite. The subroutine QRMOM can calculate for $s \geq 1$:

$$E[R^s] \quad , \quad E[R^s(a'x)] \quad , \quad E[R^s(x'Cx)],$$

where a is a vector and C a symmetric matrix, both of appropriate order. The routine is based on theory developed by Magnus (1986, 1990). The existence of the moments is also checked by QRMOM. QRMOM calls two routines from the NAG library. Users can, however, substitute their own routines if they wish.

These calculations are required in a variety of situations when econometric or time series estimators take the form of a ratio of quadratic forms in normal variables. The calculation of moments, forecast bias or mean-square forecast error relating to such estimators will involve the evaluation of expectations of one or more of the above forms. Examples are provided by the estimation of an AR(1) process with or without an intercept, stationary or non-stationary. See for example Hoque, Magnus and Pesaran (1988) and Magnus and Pesaran (1989, 1991). Further background, motivation and examples is provided in a companion paper Magnus and Pesaran (1992a). Two related subroutines, CUM and PARINT, are presented in Magnus and Pesaran (1992b). The full Fortran 77 code of QRMOM (and also of CUM and PARINT) is given in Magnus and Pesaran (1992c).

Section 2 describes QRMOM and Section 3 introduces the Fortran 77 code.

2 The Subroutine QRMOM

2.1 Description and Purpose

The subroutine QRMOM calculates the exact moments of a ratio of two quadratic forms $x'Ax$ and $x'Bx$ and related expecations. Letting $R = x'Ax/x'Bx$, QRMOM can handle the following three cases for $s = 1, 2, \cdots$:

$$E[R^s], \tag{1}$$

$$E[R^s(a'x)], \tag{2}$$

and

$$E[R^s(x'Cx)]. \tag{3}$$

Here x is an $n \times 1$ vector of normally distributed variables with some mean μ and a positive definite (hence non-singular) variance-covariance matrix Ω. A, B (positive semi-definite) and C are $n \times n$ symmetric matrices and a is an $n \times 1$ vector. In case (2) when $\mu = 0$ a simple symmetry argument shows that the expectation vanishes if it exists. QRMOM does not calculate the expectation in that case, but simply sets it equal to zero.

The subroutine QRMOM is based on theory developed by Magnus (1986, 1990), who showed that these expecations can all be expressed as single integrals and also worked out necessary and sufficient conditions for the existence of expectations in each of the above three cases. QRMOM checks for the existence of a requested s-th moment before carrying out the calculations and if the specified s is too high, it will adjust s downwards (if possible) so that existence of moments is assured.

2.2 Parameter Statements

The following parameters have been set in subroutines QRMOM and PARINT and function F:

NDIM = 50	The dimension of various work spaces. NDIM$\geq n$ where n is the number of observations.
ISPAR = 77 & ISDIM = 12	The dimensions of the array ISPRTN (ISPAR $\times$ ISDIM) where all possible partitions for a particular s are stored. This two dimensional array is set up by subroutine PARINT.
MAXMOM = 24	The maximum of s allowed.

If $n > 50$ is to be specified, the relevant parameter statements for NDIM should be increased accordingly.

If $s \leq 12$, there is no need to change the last three parameter values. If $12 < s \leq 24$, then both ISPAR and ISDIM should be increased. If $s > 24$, then not only ISPAR and ISDIM should be changed in the parameter statements, but also MAXMOM. Furthermore, the DATA statement in subroutine PARINT should be extended to include MAXMOM numbers. See Magnus and Pesaran (1992b) for further details.

2.3 Common Statements

Since the NAG routine D01AMF for the evaluation of the integral requires the function F(X) to have only one argument, the other arguments needed in the calculation of F(X) are passed through two labelled common areas QRREAL and QRINT.

2.4 Structure

SUBROUTINE QRMOM(ICASE, NOBS, IS1, IS2, A, B, C, ELA, IEMU, EMU, IOMEGA, OMEGA, ITEM, ISMAX, RESULT, ABSERR, IFAIL)

Formal Parameters

ICASE	integer	input:	1, 2, 3 or corresponding to (1), (2) or (3) above
NOBS	integer	input:	No of observations n
IS1	integer	input:	order of the lowest moment required
		output:	unchanged unless IS1 ≤ 0 in which case IS1 is set equal to one
IS2	integer	input:	order of the highest moment required
		output:	unchanged unless IS2>M where M=MIN(ISMAX,MAXMOM,ISDIM) in which case IS2 is set equal to M
A	real array of dimension at least NOBS × (NOBS+1)/2	input:	symmetric matrix in (1), (2) or (3). Only the lower part of A is stored as $a_{11}, a_{21}, a_{22}, a_{31}, a_{32}, a_{33}$ etc.
B	real array of dimension at least NOBS × (NOBS+1)/2	input:	symmetric positive semi-definite matrix B in (1), (2) or (3). Only the lower part of B is stored
C	real array of dimension at least NOBS × (NOBS+1)/2	input:	symmetric matrix C in (3). Only the lower part of C is stored. No need to assign values to C in cases (1) and (2) though storage should be allocated to it
ELA	real array of dimension at least NOBS	input:	a in (2). No need to assign values to a in cases (1) and (3) though storage should be allocated to it
IEMU	integer	input:	$= 0$ if $\mu = 0$ $\neq 0$ if $\mu \neq 0$
EMU	real array of dimension at least NOBS	input:	vector μ. Values required only if IEMU $\neq 0$ though storage should be allocated to it

IOMEGA	integer	input:	= -1 if L^{-1} is supplied where $\Omega = LL'$, L lower triangular = 1 if L is supplied where $\Omega = LL'$, L lower triangular = 2 if Ω is supplied
OMEGA	real array of dimension at least NOBS $\times$ (NOBS+1)/2	input:	either L or L^{-1} or Ω where only lower part of these are stored
ITEM	integer	output:	if ICASE = 1, ITEM indicates which condition in Theorem 1 of Magnus (1990) holds ie $1 \leq$ ITEM ≤ 3; if ICASE = 2, ITEM indicates which condition in Theorem 2 of Magnus (1990) holds ie $1 \leq$ ITEM ≤ 5; if ICASE = 3, ITEM indicates which condition in Theorem 3 of Magnus (1990) holds ie $1 \leq$ ITEM ≤ 7
ISMAX	integer	output:	the maximum of s in (1), (2) or (3) for which these expectations exist. ISMAX = 100 indicates that the expectation exists for every s
RESULT	real array of dimension at least IS2-IS1+1	output:	the required expectations stored as: RESULT(1) = IS1-th moment RESULT(2) = (IS1+1)th moment ...... RESULT(IS2-IS1+1) = IS2-th moment
ABSERR	real array of dimension at least IS2-IS1+1	output:	the absolute error in calculating each of the expectations stored in RESULT
IFAIL	integer	output:	a fault indicator where: 0: no error 1: NOBS > NDIM or NOBS ≤ 1 2: ICASE out of range 3: IOMEGA out of range

4: Eigenvalues of B could not be calculated

5: B is not positive semi-definite

6: B is the null matrix

7: if IOMEGA = 2, Ω not pos. definite;
if IOMEGA = 1, diagonal elements of
L not all positive;
if IOMEGA = -1, diagonal elements of
L^{-1} not all positive

8: if IOMEGA = 2 or 1, L can't be inverted;
if IOMEGA = -1, L^{-1} can't be inverted

9: eigenvalues of $L'BL$ could not
be calculated

10: $L'BL$ is not pos. semi-definite

11: $L'BL$ is the null matrix

12: IS1> IS2 or moments in the
adjusted range do not exist or ISDIM
in the parameter statement
is too small

13: ISPAR in the parameter statement
is too small

14-19: error in calculating the integral
corresponding to IFAIL = 1 to 6 in the NAG
library routine D01AMF

2.5 Auxiliary Algorithms

QRMOM uses two routines from the NAG library, namely F02ABF (calculation of eigenvalues and eigenvectors of a real symmetric matrix) and D01AMF (evaluation of a single integral). F02ABF is called by subroutine EVALUE and D01AMF is called by subroutine INTGRL. Users who wish not to use these two NAG library routines can substitute their own routines in subroutines EVALUE and INTGRL.

In addition to subroutines EVALUE and INTGRL, QRMOM calls various functions and subroutines:

(F1)	REAL*8 FUNCTION F(X):	used in calculating the integral
(F2)	FUNCTION INX(I,J):	picks out the appropriate element of a symmetric matrix stored in lower triangular form
(F3)	FUNCTION NFACT(N):	calculates N!
(S1)	SUBROUTINE POWER:	called by F(X)
(S2)	SUBROUTINE CALCRA:	called by F(X)
(S3)	SUBROUTINE INIT:	initializes all matrices and vectors and checks for the existence of expectations
(S4)	SUBROUTINE EXIST:	called by INIT
(S5)	SUBROUTINE PARINT:	constructs the matrix containing all partitions of an integer
(S6)	SUBROUTINE SEP:	decomposes A (pos. def.) into LL' (L lower triangular) and replaces A by L
(S7)	SUBROUTINE LOWINV:	inversion (in place) of a lower triangular matrix
(S8)	SUBROUTINE MULT1:	computes $C = B'AB$ (A symmetric, B lower triangular)
(S9)	SUBROUTINE MULT2:	computes $C = B'AB$ (A symmetric)
(S10)	SUBROUTINE NULL1:	checks to see if $AB = 0$ (A symmetric)
(S11)	SUBROUTINE NULL2:	checks to see if $B'AB = 0$ (A symmetric)

2.6 Constants

The DATA statement in QRMOM sets EPS = 1.0-11 as a small number. Any number with an absolute value below EPS will be treated as zero.

2.7 Precision

The version of the routines listed below is in double precision (Real*8). In order to change the program to single precision the following changes should be made:

(1) Change all IMPLICIT REAL*8 to IMPLICIT REAL*4.

(2) Change the constants in the DATA statements to single precision versions.

(3) Change DSQRT, DABS, DEXP to SQRT, ABS, EXP in the statement functions appearing in the beginning of routines F, EXIST, SEP, LOWINV, NULL1, NULL2.

2.8 Time and Accuracy

The results for typical CPU times and accuracy of calculations are reported in Magnus and Pesaran (1992a). These calculations were carried out using the VAX 6330 at the London School of Economics.

3 Fortran 77 Code

In Magnus and Pesaran (1992c) the complete Fortran 77 code of the following sixteen algorithms is presented. First, the main subroutine

SUBROUTINE QRMOM.

Then five algorithms that are specific to QRMOM:

REAL*8 FUNCTION F(X)
SUBROUTINE POWER
SUBROUTINE CALCRA
SUBROUTINE INIT
SUBROUTINE EXIST.

Then eight algorithms of independent interest:

FUNCTION INX(I,J)
FUNCTION NFACT(N)
SUBROUTINE SEP
SUBROUTINE LOWINV
SUBROUTINE MULT1
SUBROUTINE MULT2
SUBROUTINE NULL1
SUBROUTINE NULL2.

Finally two subroutines which use routines from the NAG library:

SUBROUTINE EVALUE
SUBROUTINE INTGRL.

There is one further subroutine called by QRMOM, namely PARINT. This subroutine is described in Magnus and Pesaran (1992a, b).

A diskette containing the code is available upon request from the authors.

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